Singularity of LCM matrices on GCD closed sets with 9 elements

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Abstract

In 1976 H. J. S. Smith defined an LCM matrix as follows: let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of positive integers with $x_1 < x_2 < \cdots < x_n$. The LCM matrix $[S]$ on the set S is the $n \times n$ matrix with $\mathrm{lcm}(x_i, x_j)$ as its ij entry.

During the last 30 years singularity of LCM matrices has interested many authors. In 1992 Bourque and Ligh ended up conjecturing that if the GCD closedness of the set S (which means that $\gcd(x_i, x_j) \in S$ for all $i, j \in \{1, 2, \ldots, n\}$), suffices to guarantee the invertibility of the matrix $[S]$. However, a few years later this conjecture was proven false first by Haukkanen et al. [3] and then by Hong [4]. It turned out that the conjecture holds only on GCD closed sets with at most 7 elements but not in general for larger sets. However, the given counterexamples did not give much insight on why does the conjecture fails exactly in the case when $n = 8$. This situation was improved in articles [5] and [6], where a new lattice theoretic approach is introduced (the method is based on the fact that because the set S is assumed to be GCD closed, the structure $(S, |)$ actually forms a meet semilattice). In article [7] this lattice-theoretic method is then developed even further.

Since the cases $n \leq 8$ have been thoroughly studied in the above mentioned articles, the next natural step is to apply the methods to the case $n = 9$. This was done by Altinisik and Altintas in [1] as they consider the different lattice structures of $(S, |)$ that can result as a singular LCM matrix $[S]$. However, their investigation leaves two open questions, and the main purpose of this presentation is to provide solutions to them.

Keywords

LCM matrix, Bourque-Ligh conjecture, GCD-closed set, meet semilattice, Möbius function.

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